Adaptive signal denoising by convex optimization

Dmitry Ostrovsky
Université Grenoble Alpes

University of Göttingen
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Ultimate goal

Recover a harmonic oscillation with $s \ll n$ frequencies:

$$x_t = \sum_{k=1}^{s} C_k e^{i\omega_k t}, \ t = 0, \ldots, n,$$

where $\{\omega_1, \ldots, \omega_s\} \subseteq [0, 2\pi)$ are unknown, from noisy observations

$$y_t = x_t + \sigma \xi_t, \quad \xi_t \sim \mathcal{N}(0, 1).$$
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y_t = x_t + \sigma \xi_t, \quad \xi_t \sim \mathcal{N}(0, 1).
\]

State of the art: Atomic Soft Thresholding (Tang et al., 2012) achieves the optimal risk

\[
\frac{\sigma^2 s \log(n)}{n}
\]

if freqs are \( \mathcal{O}(1/n) \)-separated.

\( : ( \) But without separation assumption, only slow rate \( \mathcal{O}(1/\sqrt{n}) \).

\( :) \) We achieve a near-optimal rate without separation assumption:

\[
\frac{\sigma^2 s^4 \log^2(n)}{n}
\].
**Goal:** recover discrete signal $x \in \mathbb{R}^n$ from a noisy observation

$$y_t = x_t + \sigma \xi_t, \quad t = 1, \ldots, n.$$ 

$\xi = (\xi_t)_{t=1}^n$ is standard Gaussian, and $x_t = f(t)$ for some $f : \mathbb{R} \to \mathbb{R}$. 
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- **Quadratic risk:**

  $$R(\hat{x}, x) := \frac{1}{n} \mathbb{E}[\|\hat{x} - x\|_2^2].$$

- We expect $R(\hat{x}, x) = \mathcal{O}(\sigma^2/n)$.
- **Linear estimators:** $\hat{x} = \Phi(y)$ for some linear operator $\Phi$. 
Example: recovery from a subspace

**Recovery of the mean**: suppose $x_t \equiv \mu$ for some $\mu \in \mathbb{R}$.

- Estimate $\mu$ from $n$ repeated observations $\Rightarrow$ empirical mean:

$$\hat{x} \equiv \frac{1}{n} \sum_{t=1}^{n} y_t.$$

Linear estimator, and $R(\hat{x}, x) = \sigma^2 / n$. 

- Equivalently, $x \in S$, 1-dimensional subspace spanned by all-ones vector.

- $\hat{x} = \text{proj}_S(y)$, and $R(\hat{x}, x) = \sigma^2 / n$ since $\text{proj}_S(\sigma \xi) \sim N(0, \sigma^2)$. 

- Works for any subspace! Suppose $x \in S$ of dimension $s$.

- As before, take $\hat{x} = \text{proj}_S(y)$, then $R(\hat{x}, x) = \sigma^2 / n$. 

Optimal risk up to a constant!
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Optimal risk up to a constant!
Optimality of linear estimators

When $x \in S$, there exists a linear $\hat{x}_S$ with a near-optimal risk. $\hat{x}_S$ is easy to construct if $S$ is known.

$\bar{R}(X) := \inf \sup R(\hat{x}, x) \leq \bar{R}_{\text{lin}}(X) := \inf \sup R(\hat{x}, x)$

When $X$ is a subspace, $\bar{R}_{\text{lin}}(X) \leq c \bar{R}(X) \Rightarrow$ we can search for a near-optimal estimator $\hat{x}_o$ among the linear ones!

• Donoho (1990): the above holds with $c = 1/2$ for quadratically convex and orthosymmetric sets, for example, ellipsoids.

• Juditsky & Nemirovski (2016): if $X$ is known, $\hat{x}_o$ can be computed by convex optimization!
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For any $X \subseteq \mathbb{R}^n$, define the minimax risk and the linear minimax risk:

$$\bar{R}(X) := \inf_{\hat{x}} \sup_{x \in X} R(\hat{x}, x) \leq \bar{R}^{\text{lin}}(X') := \inf_{\hat{x} = \Phi(y)} \sup_{x \in X} R(\hat{x}, x).$$

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If “good” $\mathcal{X}$ is unknown, $\hat{x}^o$ still exists, but not accessible directly.

- For example, $x \in \{\mathcal{X}_\alpha\}$, large family of “good” sets (subspaces).

**Question:** *Is it possible to “mimick” $\hat{x}^o$, i.e. construct an adaptive estimator $\hat{x} = \hat{x}(y)$ with a comparable risk?*
Adaptive estimation

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**Question:** Is it possible to “mimick” $\hat{x}^o$, i.e. construct an **adaptive estimator** $\hat{x} = \hat{x}(y)$ with a comparable risk?

- Adaptive estimator $\hat{x}$ approaches $R(\hat{x}^o, x)$ without knowing $x$:

$$R(\hat{x}, x) \approx R(\hat{x}^o, x).$$

- We hope to find such $\hat{x}$ by a data-driven (and efficient) search over a class of linear estimators.
Filters

In signal processing, we usually assume **time-invariance** of some kind. Recall that we estimate the signal on the regular grid:

\[ y_t = x_t + \sigma \xi_t, \quad t \in \{ -n, \ldots, 0, \ldots, n \}. \]

- Consider **time-invariant linear estimators**: convolution of \( y \) with a filter \( \varphi \in B_m = \{ \text{“vanish outside \([0, m]\) for some } m \leq n \} \):

\[
\hat{x}_t = [\varphi \ast y]_t := \sum_{\tau=0}^{m} \varphi_{\tau} y_{t-\tau}, \quad t \in [-n + m, n].
\]
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\[ \hat{x}_t = [\varphi \ast y]_t := \sum_{\tau=0}^{m} \varphi_\tau y_{t-\tau}, \quad t \in [-n + m, n]. \]

- **Goal**: recovery on \([0, n]\) via previous observations, with the risk

\[ R_n(\varphi, x) := \frac{1}{n} \mathbb{E}[\| [x - \varphi \ast y]_0^n \|_2^2], \]

where \([x]_a^b = [x_a, \ldots, x_b]\).
Main assumption: LTI recoverability

We assume that the class of linear filtering estimators is powerful.

**Definition.** $x$ is $\varrho$-recoverable if there exists a $\phi^o \in B_{n/2}$ satisfying

$$R_n(\phi^o, x) \leq \frac{\sigma^2 \varrho}{n}.$$ 

Adaptive signal denoising: find $\hat{\phi} = \hat{\phi}(y)$ s.t. $R_n(\hat{\phi}, x) \approx R_n(\phi^o, x)$. 

![Diagram showing n/2]
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**Bias-variance decomposition:**

$$\frac{1}{n} \mathbb{E} \left[ \left\| x - \phi^o \ast y \right\|_2^2 \right] = \frac{1}{n} \left\| x - \phi^o \ast x \right\|_2^2 + \frac{\sigma^2}{n} \mathbb{E} \left[ \left\| \phi^o \ast \xi \right\|_2^2 \right]$$

- reproduction of the signal: $\frac{1}{n} \left\| x - \phi^o \ast x \right\|_2^2 \leq \frac{\sigma^2 \varrho}{n}$,
- small $\ell_2$-norm of the oracle: $\left\| \phi^o \right\|_2 \leq \frac{\varrho}{n}$. 
Adaptive estimator

Let \( \mathcal{F} \) be the Discrete Fourier transform operator on \([0, n]\):

\[
\mathcal{F}_{j\tau} = \frac{1}{\sqrt{n+1}} \exp \left( \frac{2\pi ij\tau}{n+1} \right), \quad 0 \leq j, \tau \leq n.
\]

We propose an adaptive estimator: \( \hat{x} = \hat{\varphi} * y \), where \( \hat{\varphi} \in B_n \) is

\[
\hat{\varphi} \in \text{argmin}_{\varphi \in B_n} \left\{ \left[ (y - \varphi * y)_{0} \right]_0^2 : \| \mathcal{F} \varphi \|_1 \leq \varrho / \sqrt{n} \right\} 
\]

Compare with the spectral Lasso:

\[
\hat{x} \in \text{argmin}_{x \in \mathbb{R}^n} \left\{ \left[ y - x_{0} \right]_0^2 : \| \mathcal{F} x \|_1 \leq \| \mathcal{F} x^o \|_1 \right\}.
\]

- No sparsity. The “dictionary matrix” \( Y \) s.t. \( \varphi * y = Y(\mathcal{F} \varphi) \) is not RIP and scales differently with \( \sigma \). Standard techniques fail.
Recall $\varrho$-recoverability of $x$: there exists a $\phi^o \in B_{n/2}$ such that

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Statistical bound

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**Theorem** (Main Result)

If $x$ is $\varrho$-recoverable, the filter $\hat{\phi}$ satisfies

$$R_n(\hat{\phi}, x) \leq \frac{\sigma^2 \varrho}{n} (\varrho + \log n).$$

(actually a bound w.h.p.)

Price of adaptation is $\varrho \Rightarrow$ we would like $\varrho$ to be as small as possible.
Statistical bound: naive approach

- There exists a $\phi^o \in B_{n/2}$ for which $\|\phi^o\|_2^2 \leq \frac{\varrho}{n}$, $R_n(\phi^o, x) \leq \frac{\sigma^2 \varrho}{n}$.
- Suppose that $\varrho$ is known, and search for $\phi^o$:
  $$\hat{\phi} \in \arg\min_{\phi \in B_{n/2}} \left\{ \frac{1}{n} \| [y - \phi^* y]^n \|_2^2 : \|\phi\|_2^2 \leq \frac{\varrho}{n} \right\}.$$
- $\phi^o$ is feasible, so that
  $$\frac{1}{n} \| y - \hat{\phi}^* y \|_2^2 \leq \frac{1}{n} \| y - \phi^o \|_2^2 = R_n(\phi^o, x) + \frac{\sigma^2}{n} \| \xi \|_2^2 + \langle \ldots \rangle.$$
- OK at this step: $Q_n(\phi^o, x)$ is small, $\sigma^2 \| \xi \|_2^2$ subtracted. **But:**
  $$\frac{1}{n} \| x - \hat{\phi}^* y \|_2^2 = \frac{1}{n} \| y - \hat{\phi}^* y \|_2^2 - \frac{\sigma^2}{n} \| \xi \|_2^2 + \langle \ldots \rangle + \frac{2\sigma^2}{n} \langle \xi, \hat{\phi}^* \xi \rangle.$$

$\ell_2$-constraint too weak to control $\langle \xi, \hat{\phi}^* \xi \rangle$ because $\hat{\phi}$ is random.
There exists a $\phi^o \in B_{n/2}$ for which $\|\phi^o\|_2^2 \leq \frac{\varrho}{n}$, $R_n(\phi^o, x) \leq \frac{\sigma^2 \varrho}{n}$.

Instead of $\phi^o$, let’s mimic $\varphi^o := (\phi^o * \phi^o) \in B_n$. Can show:

$$\|\mathcal{F}\varphi^o\|_1^2 \leq \frac{\varrho^2}{n},$$

$$R_n(\varphi^o, x) \leq \frac{\sigma^2 \varrho^2}{n}.$$
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$$\|F \varphi^o\|_1^2 \leq \frac{\varrho^2}{n},$$

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Pay an extra $\varrho$, but obtain a bound on the $\ell_1$-norm (in Fourier).

Problem term $\langle \xi, \hat{\varphi} \ast \xi \rangle$: uniform bound + extreme points.

Adaptive estimator $\hat{\varphi}$ can be formulated as

$$\hat{\varphi} \in \arg\min_{\varphi \in B_n} \left\{ \frac{1}{n} \|y - \varphi \ast y\|_2^2 : \|F \varphi\|_1 \leq \frac{\varrho}{\sqrt{n}} \right\}$$

or the penalized problem (useful when $\varrho$ is unknown).
**Definition.** Subspace $\mathcal{S}$ of the space of sequences $(\ldots, x_{-1}, x_0, x_1, \ldots)$ is called **time-invariant** if it is preserved under $x_t \mapsto x_{t-1}$. 

**Time-Invariant Subspace Assumption (TISA):** $x$ belongs to some time-invariant subspace of dimension $s \leq n$. 

$\text{TISA} \iff \text{exp. polynomials.}$ 

$x$ satisfying TISA is an exponential polynomial of order $s$, with frequencies depending on $S$. 

**Example:** harmonic oscillation $x_t = \sum_{k=1}^{s} C_k e^{i\omega_k t}$, $\tau \in \mathbb{Z}$. 


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x_t = \sum_{k=1}^{s} C_k e^{i \omega_k t}, \quad \tau \in \mathbb{Z}.
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Lower bound: $\varrho(s) = s$. Achievable if we allow for bilateral filters:
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Theorem
Let $x$ satisfy TISA with some $s \leq n$. Then, $x$ is $\varrho$-recoverable, with respect to bilateral oracle, with $\varrho = s$. 
Denoising harmonic oscillations

**Goal:** recover \( x \) on \([-n, n]\) when frequencies are unknown:

\[
x_\tau = \sum_{k=1}^{s} C_k e^{i\omega_k \tau},
\]

Atomic Soft Thresholding (Tang & Recht, 2012):

\[
R_n \leq \frac{\sigma^2 s \log n}{n}
\]

if frequencies are separated, but slow rate \( O(1/\sqrt{n}) \) if not.

**Adaptive filtering:**

\[
R_n \leq \frac{\sigma^2 s^4 \log^2 n}{n}
\]

without any separation assumptions. \( s^4 \) improves to \( s^2 \):

- in the separated case via Beurling’s majorant (Moitra, 2014).
- in the central zone \([-n/2, n/2]\) via bilateral filters.
Optimization problem

For some $r > 0$, we want to solve:

$$\text{Opt} = \min_{\varphi \in \mathbb{C}^n} \left\{ f(\varphi) = \|y - y * \varphi\|_2^2 : \|F_n\varphi\|_1 \leq r \right\}. \quad (P)$$

- Well-structured feasible set – $\ell_2/\ell_1$-norm ball, prox in $O(n \log n)$.
- First-order oracle can be computed in $O(n \log n)$.
- Low-accuracy solutions: sufficient to find a solution $\tilde{\varphi}$ satisfying

$$\varepsilon(\tilde{\varphi}) := f(\tilde{\varphi}) - \text{Opt} \lesssim \frac{1}{n} \text{Opt}.$$

$\Rightarrow$ proximal gradient methods.
Change of variables

\[
\text{Opt} = \min_{\varphi \in \mathbb{C}^n} \left\{ f(\varphi) = \|y - y \ast \varphi\|_2^2 : \|F_n \varphi\|_1 \leq r \right\}. \quad (P)
\]

\[u := \frac{F_n(\varphi)}{r} \Rightarrow \text{feasible set is the unit ball of the (complex) } \ell_1\text{-norm.}\]

\[y \ast \varphi = y \ast F_n^{-1}(ru)
= F_n^{-1} \left\{ F_{3n}[y;0_n] \bullet F_{3n}[0_{2n};F_n^{-1}(ru)] \right\} = Au,
\]

where \([x;0_n]\) is the concatenation with the zero vector of length \(n\), and \(\bullet\) is the element-wise product. Computed in \(\mathcal{O}(n \log n)\) by FFT.

\[f(\varphi) = F(u) = \|y\|_2^2 - \langle y, Au \rangle - \langle Au, y \rangle + \langle u, A^T Au \rangle,
\]
\[\nabla F(u) = 2(-A^T y + A^T Au)\]

(everything is complex-valued, hiding some conjugates).
Proximal mapping

So, now \((P)\) is reformulated as a well-structured optimization problem

\[
\text{Opt} = \min_{u \in \mathbb{C}^n} \left\{ F(u) : \|u\|_1 \leq 1 \right\} , \tag{P'}
\]

where we can compute \(F(u)\) and \(\nabla F(u)\) in \(O(n \log n)\).

We also must be able to compute the proximal mapping:

\[
\text{prox}_u \left( g \right) := \arg\min_{\|v\|_1 \leq 1} \left\{ \langle g, v \rangle + D_u(v) \right\} ,
\]

where

\[
D_u(v) := \omega(v) - \omega(u) - \langle \nabla \omega(u), v - u \rangle
\]

is the Bregman divergence, and \(\omega(u)\) is a “good” proximal function: smooth, 1-strongly convex, with computable prox, and with a small\n
\[
R^2 = \max_{\|u\|_1 \leq 1} \omega(u).
\]
Proximal functions

Euclidean prox:

$$\omega(u) = \frac{1}{2} \|u\|_2^2 \quad \Rightarrow \quad D_u(v) = \frac{1}{2} \|v - u\|_2^2.$$ 

Corresponding prox is Euclidean projection on the complex $\ell_1$-ball.

- Computable in $O(n \log n)$, $R^2 = O(1)$.
- Smoothness measured in $\ell_2$-norm.

“Suitable” prox:

$$\omega(u) = \gamma \|u\|_p^p, \quad p = 1 + \frac{1}{\ln n}, \quad \gamma = \frac{e \ln n}{p}.$$ 

- Computable in $O(n \log n)$, $R^2 = O(\log n)$.
- Smoothness measured in $\ell_q$-norm, $q \approx \log n \quad \Rightarrow \quad \| \cdot \|_q \leq C \| \cdot \|_\infty$. 
Solving the optimization problem

Let \( L \) be the Lipschitz constant of \( \nabla F(u) \) (precomputed from data).

**Fast Gradient Method** (Nesterov & Nemirovski, 2013)

**Initialization:** \( u_0 = 0; \ G_0 = 0. \)

**For** \( t = 0, 1, \ldots \) **do**

(a) \( w_t = \text{prox}_0 \left( \frac{G_t}{L} \right) \).

(b) \( \tau_t := \frac{2(t+2)}{(t+1)(t+4)} \).

(c) \( v_{t+1} := \tau_t w_t + (1 - \tau_t) u_t \)

(d) \( \hat{v}_{t+1} := \text{prox}_{\frac{\tau_t}{L}} \left( \frac{t+2}{2} \frac{\nabla F(v_{t+1})}{L} \right) \).

(e) \( u_{t+1} := \tau_t \hat{v}_{t+1} + (1 - \tau_t) u_t, \ G_{t+1} := G_t + \frac{t+2}{2} \nabla F(v_{t+1}) \)

Similar to Fast Gradient Descent. **Convergence guarantee:**

\[
F(u_t) - F^* \lesssim \frac{LR^2}{t^2}
\]
Experiments

Figure: Signal and image denoising in different scenarios, 1-d (left) and 2-d (right).
Demonstration

Brodatz D75, SNR=1. Similar MSE, but Lasso tends to over-smooth.
We give an efficiently computable and statistically near-optimal construction of adaptive estimator for time-invariant signals.

Main idea: adaptation to the well-performing linear estimator.

As a consequence, we get fast rates of denoising harmonic oscillations without the frequency separation assumption.

Thank you for your attention!
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Collaborators

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Publications

Adaptive estimation: classical example

Suppose $x$ is $s$-sparse, i.e. comes from $S$ spanned by $\{e_{i_1}, \ldots, e_{i_s}\}$.

- Linear oracle $\hat{x}^o = \text{proj}_S(y)$:
  \[
  Q(\hat{x}^o, x) = \frac{\sigma^2 s}{n}
  \]

- Soft-thresholding estimator (Lasso):
  \[
  \hat{x} = \arg\min_{x \in \mathbb{R}^n} \left\{ \|x - y\|_2^2 + \lambda \|x\|_1 \right\}.
  \tag{1}
  \]

  If $\lambda$ is well-chosen, $\hat{x}$ is adaptive: not knowing $S$, it satisfies
  \[
  Q(\hat{x}, x) \leq Q(\hat{x}^o, x) \log(n),
  \]

- $\hat{x}$ is non-linear but “looks” like a linear estimator, and can be computed by searching over linear estimators!
- Indeed, (1) is separable, and we can write $\hat{x} = \hat{\varphi} \cdot y$, where
  \[
  \hat{\varphi} = \arg\min_{\varphi \in \mathbb{R}^n} \left\{ f_y(\varphi) := \|y - y \cdot \varphi\|_2^2 + \lambda \|y \cdot \varphi\|_1 \right\}.
  \]
Better complexity estimate

After \( k \) iterations of FGM, we have for \((P^2)\):

\[
f^2(\varphi_k) \leq \text{Opt}^2 + \frac{LR^2}{k^2}.
\]

We get \( \mathcal{O}(k^{-1}) \) error for the initial problem \((P)\):

\[
f(\varphi_k) \leq \text{Opt} + \frac{\sqrt{LR}}{k}.
\]

Additional structure: since \( \text{Opt} \geq 0 \),

\[
f^2(\varphi_k) - \text{Opt}^2 = (f(\varphi_k) - \text{Opt})(f(\varphi_k) + \text{Opt}) \geq 2\text{Opt}(f(\varphi_k) - \text{Opt}),
\]

and we get an “optimistic” \( \mathcal{O}(k^{-2}) \) error provided that \( \text{Opt} > 0 \):

\[
f(\varphi_k) - \text{Opt} \leq \frac{LR^2}{2\text{Opt}k^2}
\]